UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE

UMENTATION PAGE

Form Approved OMB No. 0704-0188

AD-A221 253

1b. RESTRICTIVE MARKINGS

2b. DECLASSIFICATION / DOWNGRADING SCHEDULE
N/A

3. DISTRIBUTION / AVAILABILITY OF REPORT
Approved for Public Release;
Distribution Unlimited

4. PERFORMING ORGANIZATION REPORT NUMBER(S)
Technical Report No. 275

5. MONITORING ORGANIZATION REPORT NUMBER(S)
[illegible]OSR·TR· 9[illegible]398

6a. NAME OF PERFORMING ORGANIZATION
University of North Carolina
Center for Stochastic Processes

6b. OFFICE SYMBOL *(If applicable)*

7a. NAME OF MONITORING ORGANIZATION
AFOSR/NM

6c. ADDRESS *(City, State, and ZIP Code)*
Statistics Department
CB #3260, Phillips Hall
Chapel Hill, NC 27599-3260

7b. ADDRESS *(City, State, and ZIP Code)*
Bldg. 410
Bolling Air Force Base, DC 20332-6448

8a. NAME OF FUNDING / SPONSORING ORGANIZATION
AFOSR

8b. OFFICE SYMBOL *(If applicable)*
NM

9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER
F49620 85C 0144

8c. ADDRESS *(City, State, and ZIP Code)*
Bldg. 410
Bolling AFB, DC 20332-6448

10. SOURCE OF FUNDING NUMBERS

PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO.	WORK UNIT ACCESSION NO.
6.1102F	2304	A5	

11. TITLE *(Include Security Classification)*
A white noise theory of infinite dimensional calculus

12. PERSONAL AUTHOR(S)
Hida, T.

13a. TYPE OF REPORT
preprint

13b. TIME COVERED
FROM ________ TO ________

14. DATE OF REPORT *(Year, Month, Day)*
1989, Oct.

15. PAGE COUNT
26

16. SUPPLEMENTARY NOTATION
None.

17. COSATI CODES

FIELD	GROUP	SUB-GROUP
XXXXXX	XXXXXXXXXXX	XXX

18. SUBJECT TERMS *(Continue on reverse if necessary and identify by block number)*
None.

19. ABSTRACT *(Continue on reverse if necessary and identify by block number)*

Sections 1-4 are based on those three lectures with somewhat more attention devoted to the space of generalized white noise functionals. What is described here are mostly survey articles, though some state-of-the-art results are added, while Section 5 involves a new approach to the study of Gaussian random fields. This topic is exactly what I wished to propose at the colloquium. What is going to be presented here is, of course, far from a general theory; however it is our hope that this attempt would be the very first step towards the study of Gaussian random fields using variational calculus.

20. DISTRIBUTION / AVAILABILITY OF ABSTRACT
☒ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT. ☐ DTIC USERS

21. ABSTRACT SECURITY CLASSIFICATION
UNCLASSIFIED

22a. NAME OF RESPONSIBLE INDIVIDUAL
~~Professor Eytan Barouch~~ Dr. Jon A Sjogren

22b. TELEPHONE *(Include Area Code)*
(202)767-~~5026~~ 4940

22c. OFFICE SYMBOL
AFOSR/NM

DD Form 1473, JUN 86 — Previous editions are obsolete.

SECURITY CLASSIFICATION OF THIS PAGE

UNCLASSIFED

CENTER FOR STOCHASTIC PROCESSES

Department of Statistics
University of North Carolina
Chapel Hill, North Carolina

A WHITE NOISE THEORY OF INFINITE DIMENSIONAL CALCULUS

by

Takeyuki Hida

Technical Report No. 275

October 1989

A WHITE NOISE THEORY OF INFINITE DIMENSIONAL CALCULUS

Takeyuki Hida

Department of Mathematics
Nagoya University

and

Center for Stochastic Processes
University of North Carolina at Chapel Hill

<u>Preface</u> These are notes of my three lectures for graduate students and a colloquium talk at the Department of Statistics, University of North Carolina, Chapel Hill, in April 1989.

Sections 1-4 are based on those three lectures with somewhat more attention devoted to the space of generalized white noise functionals. What is described here are mostly survey articles, though some state-of-the-art results are added, while Section 5 involves a new approach to the study of Gaussian random fields. This topic is exactly what I wished to propose at the colloquium. What is going to be presented here is, of course, far from a general theory; however it is our hope that this attempt would be the very first step towards the study of Gaussian random fields using variational calculus.

I am grateful to the Center for Stochastic Processes, in particular, to Professor G. Kallianpur who has given me this opportunity to give lectures and has suggested to write these notes.

Contents

Research supported by the Air Force Office of Scientific Research Contract F49620 85C 0144.

§1. White noise

We start with a Gel'fand triple

$$E \subset L^2(\mathbb{R}^d) \subset E^*, \quad d \geq 1,$$

where E is a real nuclear space of E. These spaces E and E^* are linked by the canonical bilinear form $\langle \ , \ \rangle$.

Given a characteristic functional

(1.1) $$C(\xi) = \exp[-½\|\xi\|^2], \quad \xi \in E, \quad \| \ \|: L^2(\mathbb{R}^d)\text{-norm}.$$

We can form a p¬obability measure μ on E^* such that

(1.2) $$C(\xi) = \int_{E^*} \exp[i \langle x, \xi \rangle] \, d\mu(x).$$

The probability measure μ is viewed as the probability distribution of white noise $W(u)$, $u \in \mathbb{R}$, which is a stationary generalized Gaussian random field with independent values at every point $u \in \mathbb{R}^d$. Thus, the measure space (E^*, μ) is a realization of white noise and μ-almost all x in E^* is thought of as a sample function. We call the probability measure space white noise and μ is said to be the measure of white noise.

Remark 1. As for the story behind the definition of white noise the reader is recommended to see [2, Chapter 1].

Let $\xi \in E$ be fixed. Then, a linear functional $\langle x, \xi \rangle$ of $x \in E^*$ is a random variables on (E^*, μ). If $\{\xi_n\}$ is an orthonormal system in $L^2(\mathbb{R}^d)$, then $\{\langle x, \xi_n \rangle\}$ forms a system of independent identically distributed (i.i.d.) random variables, of course standard Gaussian in distribution.

Remark 2. Let $\{\langle x, \xi_n \rangle\}$ be the system given above. Then, $\{\langle x, \xi_n \rangle^2\}$ is also an i.i.d. system with common unit mean value. We can therefore appeal to the strong law of large numbers to obtain

$$\frac{1}{N}\sum_{1}^{N} \langle x,\xi_n\rangle^2 \longrightarrow 1, \qquad \text{a.e.} \tag{1.3}$$

Since $\{\langle x,\xi_n\rangle\}$ looks like a coordinate system for x in E^*, the formula (1.3) tells us that μ-almost all x is sitting on, as it were, an infinite dimensional sphere with radius $\sqrt{\infty}$. Such an intuitive observation leads us to introduce an infinite dimensional rotation group and even to discuss so-to-speak harmonic analysis arising from the rotation group.

We now come to the complex Hilbert space

$$(L^2) = L^2(E^*,\mu).$$

The Wiener-Itô decomposition of (L^2), which is well known, may be obtained in the following manner. Let $\{\xi_n\}$ be a complete orthonormal system (CONS) in $L^2(R^1)$ such that $\xi_n \in E$ for every n. Then, it is proved that the set of the Fourier-Hermite polynomials of the form

$$C_{\{n_k\}}\Pi_k\, H_{n_k}(\langle x,\xi_k\rangle/\sqrt{2}), \qquad \text{(finite product)}, \tag{1.4}$$

$$C_{\{n_k\}} = (\Pi_k\, n_k!2^{n_k})^{\frac{1}{2}}, \quad \Sigma n_k = \text{degree},$$

forms a CONS in (L^2). The subset of those polynomials of degree n spans a closed subspace $\mathcal{H}_n$ of (L^2). Obviously, $\mathcal{H}_n$'s are mutually orthogonal. We can further prove

Theorem 1.1 (Wiener-Itô) The space (L^2) admits a direct sum decomposition in terms of $\mathcal{H}_n$:

$$(L^2) = \oplus_{n=0}^{\infty} \mathcal{H}_n. \tag{1.5}$$

A member of $\mathcal{H}_n$ is called a multiple Wiener integral of degree n. The space (L^2) with this decomposition is often called a <u>Fock space</u>.

We now introduce the $\mathcal{T}$-transform

$$(\mathcal{T}\varphi)(\xi) = \int_{E^*} \exp[i\langle x,\xi\rangle]\varphi(x)d\mu(x), \quad \varphi(x) \in (L^2). \tag{1.6}$$

The vector space $\mathcal{F} = \{\mathcal{T}\varphi\colon \varphi \in (L^2)\}$ can be topologized so as to be a reproducing kernel Hilbert space with kernel $C(\xi-\eta)$, $(\xi,\eta) \in E\times E$, where C is the characteristic functional given by (1.1). We can prove

<u>Theorem 1.2</u>. The $\mathcal{T}$-transform gives an isomorphism

$$(L^2) \cong \mathcal{F}.$$

A visualized expression of an $\mathcal{H}_n$-functional can be obtained from the following theorem.

<u>Theorem 1.3</u> (Integral representation) For $\varphi(x)$ in $\mathcal{H}_n$, we have

$$(\mathcal{T}\varphi)(\xi) = i^{\eta}C(\xi)U(\xi). \tag{1.7}$$

where U is expressed in the form

$$U(\xi) = \int_{\mathbb{R}^n} F(u)\xi^{n\otimes}(u)d^n u, \quad F \in L^2(\mathbb{R}^n) = L^2(\mathbb{R}^{\cdot})^{\hat{n\otimes}} \tag{1.8}$$

with the properties that

$$F \longleftrightarrow \widehat{L^2}(\mathbb{R}^n), \quad \text{bijective}$$

$$\|\varphi\| = \sqrt{n!}\ \|F\|_{\widehat{L^2}(\mathbb{R}^n)}.$$

This result may be expressed briefly as

$$(1.8)\qquad \begin{cases} \mathcal{H}_n \cong \widehat{L^2}(R^n, n!d^n u), & \text{and as} \\ (L^2) \cong \bigoplus_{n=0}^{\infty} \widehat{L^2}(R^n, n!d^n u). \end{cases}$$

The functional $U(\xi)$ in (1.8) may be obtained as the <u>$\mathcal{S}$-transform</u>:

$$(1.9)\qquad (\mathcal{S}\varphi)(\xi) = U(\xi) = \int_{E^*} \varphi(x+\xi)d\mu(x).$$

We then provide a powerful tool for our analysis, namely an infinite dimensional rotation group (cf. Remark 2). Set

$$O(E) = \{g;\ \text{i)}\ g \text{ is a linear homeomorphism of } E, \text{ and}$$
$$\text{ii)}\ \|g\xi\| = \|\xi\| \quad \text{for any } \xi \in E\}.$$

Obviously, $O(E)$ forms a group under the usual product which we may even topologize, say by the compact-open topology, so that it is a topological group.

<u>Definition</u>. The group $O(E)$ is called a <u>rotation group</u> of E, or often called an <u>infinite dimensional rotation group</u> and denoted by $O(\infty)$, when E need not be specified.

Associated with each g is the adjoint g^* which is defined in the usual manner:

$$\langle g^* x, \xi\rangle = \langle x, g\xi\rangle, \quad x \in E^*,\ \xi \in E.$$

<u>Theorem 1.4</u>. For any g in $O(E)$

$$(1.10)\qquad g^*\mu = \mu$$

holds.

<u>Proof</u>. The characteristic functional of the measure $g^*\mu$ is $C(g^{-1}\xi)$ which

is in agreement with the characteristic functional $C(\xi)$ of μ. This ensures (1.10).

We now pause to explain some aspects of our infinite dimensional calculus in order to describe the idea of our approach.

I. We have been motivated by P. Lévy's functional analysis (see [1]) where he has discussed the analysis of functionals on $L^2([0,1])$. The attempt to introduce a uniform measure on the unit ball never succeeds, as he knew and we now know, but he has introduced the notion of "la valeur moyenne" of functionals on the unit ball while we have been led to use the (real, countably additive) measure not on $L^2([0,1])$ but on the space E^* of generalized functions. Still we can find lots of ideas from Lévy's results when we proceed to work out a calculus of white noise functionals.

II. Since we have the rotation group acting on an "infinite dimensional sphere", we may see a counterpart of the analysis on finite dimensional spheres. The unitary representation theory of Lie groups has given us valuable suggestions. The approach in this line is quite successful. In addition, we can see several profound and in fact essentially infinite dimensional properties of white noise functionals through subgroups of $O(E)$ that can not be approximated by finite dimensional rotations. This will be seen in Section 3.

For the details of what has been discussed in this section, we refer to the book [3].

§2. Generalized functionals.

We are going to carry out the so-called causal calculus, where the development of time t is involved explicitly and where $\{x(t): t \in \mathbb{R}^1\}$ for x in E is taken to be the system of variables of a white noise functional. It is, therefore, quite reasonable to introduce some classes of generalized functionals which involve, e.g., like polynomials in x(t)'s, exponential functions and even delta-functions of those variables. We are going to introduce the following two classes of generalized white noise functionals, each of which plays its own role in our calculus. Several variants may, of course, be considered depending on the purposes. However we shall introduce only generic cases.

[I] Spaces $(L^2)^{\pm}$.

We start with the Fock space established in (1.5). Take a Sobolev space $H^{a_n}(R^n)$ of order $a_n > 0$. Set $\widehat{H^{a_n}}(R^n) = H^{a_n}(R^n) \cap \widehat{L^2}(R^n)$. Then the isomorphism (1.8) can be restricted to $H^{a_n}(R^n)$ to define $\mathcal{H}_n^{(n)} \subset \mathcal{H}_n$. The dual space $\mathcal{H}_n^{(-n)}$ of $\mathcal{H}_n^{(n)}$ can be obtained and we establish the following diagram:

$$\begin{aligned} &\mathcal{H}_n^{(-n)} \cong \hat{H}^{-(n+1)/2}(R^n) \\ &\quad\cup \\ &\mathcal{H}_n \cong \widehat{L^2}(R^n) \\ &\quad\cup \\ &\mathcal{H}_n^{(n)} \cong \widehat{H}^{(n+1)/2}(R^n) \qquad \text{(up to const. } \sqrt{n!}\text{)} \end{aligned}$$

Set

$$(L^2)^+ = \{\varphi = \sum_n \varphi_n : \varphi_n \in \mathcal{H}_n^{(n)},\ \Sigma c_n^2 \|\varphi\|_n^2 = \|\varphi\|_\infty^2 < \infty\}$$

where $\{c_n\}$ is an increasing sequence of positive numbers, and where $\|\ \|_n$ is the $\mathcal{H}_n^{(n)}$-norm. The $\|\ \|_\infty$ appearing above is a Hilbertian norm, with respect to which $(L^2)^+$ becomes a Hilbert space.

Let $(L^2)^-$ be the dual space of $(L^2)^+$. A member of $(L^2)^-$ is called a <u>generalized white noise functional</u>. Because of the construction of $(L^2)^+$, we often use the following notation:

$$(L^2)^- = \bigoplus_{n=0}^{\infty} c_n^{-1} \mathcal{H}_n^{(-n)}, \tag{2.1}$$

and the canonical bilinear form which connects $(L^2)^+$ and $(L^2)^-$ is denoted by $\langle\ ,\ \rangle_\mu$. If no confusion occurs, we denote it simply as $\langle\ ,\ \rangle$.

Remark 1. One may ask why the choice of $a_n = (n+1)/2$ is most acceptable. There are many reasons. For one thing, the kernel function of the integral representation of $\mathcal{H}_n$-functional has a continuous version and its restriction to a lower, say d-dimensional space is again in the Sobolev space with the order satisfying the same relation to the dimension: $(d+1)/2$. For another reason, we can claim that Hermite polynomials in $x(t)$'s of degree n are living in the space $\mathcal{H}_n$.

Examples of an $(L^2)^-$ - functional.

1°) Hermite polynomial in $x(t)$ of degree n will be denoted by $:x(t)^n:$. The polynomial as well as an integral of the form

$$\int f(u):x(u)^n:\,du, \quad f \in L^2(R),$$

belong to $\mathcal{H}_n$, since $\delta_t^{\otimes n}$ is in $H^{-(n+1)/2}(R^n)$.

2°) An exponential function formally given by

$$\varphi_c(x) = \exp[c \int_T x(t)^2 dt], \ c \in \mathbb{C} \quad \mathrm{Re}\, c < \tfrac{1}{2}, \ T \text{ interval},$$

has no meaning, but applying the multiplicative renormalization such as

$$\tilde{\varphi}_c(x) = \mathcal{N} \exp[c \int_T x(t)^2 dt], \quad \mathcal{N} \text{ normalizing factor}$$

is a member of $(L^2)^-$. Its $\mathcal{S}$-transform is given by

$$\exp[\tilde{c} \int_T \xi(t)^2 dt], \quad \tilde{c} = c/(1-2c).$$

3°) A normal functional (named by P. Lévy) is an $(L^2)^-$ - functional with the $\mathcal{S}$-transform expres ible as

$$U(\xi) = \sum_{\{n_j\}} \int \ldots \int F_{\{n_j\}}(u_1, \ldots, u_k)\xi(u_1)^{n_1} \ldots \xi(u_k)^{n_k} du_1 \ldots du_k .$$

is also a member of $(L^2)^-$ with suitable assumptions on $F_{\{n_j\}}$

4°) $(L^2)^-$ functionals related to the delta function.

Donsker's delta function

$$\delta_{t,y}(x) = \delta_o(y-B(t,x)),$$

where $B(t,x)$ is a Brownian motion formed on (E^*, μ). $(L^2)^-$ - functional given by Kallianpur and Kuo.

$$f \circ B(t,x) = \int f(y)\delta(y - B(t,x))dy,$$

[II] Spaces (S) and $(S)^*$.

We use the second quantization technique to introduce the test functional space (S). There is a somewhat general theory (see the book [5]), where we start with a σ-Hilbert nuclear space and lift up the structure to Fock spaces. However, to concretize the story we shall form the space by using concrete well known spaces and operators. The basic nuclear space is now taken to be the Schwartz space $\mathcal{S}(R^1)$, which is the <u>core</u> in terms of Glimm-Jaffe. Let A be given by

$$A = -\frac{d^2}{du^2} + u^2 + 1 \tag{2.1}$$

which is positive and self-adjoint. Its domain is taken to be $\mathcal{S}(R^1)$. Then, there is the second quantized operator

(2.2)
$$\Gamma(A) = \bigotimes_n A^{\otimes n}$$

acting on the Fock space formed from $L^2(R^1)$ as we did in (1.5). It holds that

$$\Gamma(A)^p = \Gamma(A^p), \quad p \in Z_+ .$$

By the isomorphism (1.8) we can see that the $\Gamma(A)^p$ goes to an operator acting on (L^2). For simplicity we shall denote this operator on (L^2) by the same symbol $\Gamma(A)^p$.

Now set

$$(S_p) = \mathfrak{D}(\Gamma(A^p))$$

and denote by (S_{-p}) the dual space of (S_p). Then we obtain a chain of the spaces

$$\ldots \subset (S_{p+1}) \subset (S_p) \subset \ldots \subset (L^2) \subset \ldots \subset (S_{-p}) \subset (S_{-p-1}) \subset \ldots$$

Let $\| \ \|_k$ be the norm in the Hilbert space (S_k). Then, $\{\| \ \|_k ; k \in Z\}$ is compatible in the sense of Gel'fand-Vilenkin. Now recall that the Hermite functions (products of Hermite polynomials and Gaussian kernel with normalizing constant) ξ_k are the eigenfunctions of the operator A:

(2.3)
$$A\xi_k = (2k+2)\xi_k .$$

With these properties in mind we can easily prove

Proposition 2.1. i) The injection

$$(S_{k+1}) \longrightarrow (S_k)$$

is of Hilbert-Schmidt type.

Set

$$(S) = \bigcap_p (S_p)$$

and let the projective limit topology be provided for (S).

Proposition 2.2

i) The space (S) is nuclear.

ii) (S) is an algebra.

iii) (S) is dense in (L^2).

The dual $(S)^*$ of (S) is therefore given by

$$(S)^* = \bigcup_p (S_{-p})$$

A member of $(S)^*$ is also called a generalized white noise functional.

Examples. 1) An exponential function of the form $\exp[c\langle x,\xi\rangle]$, $c \in \mathbb{C}$ are members of (S).

2) $\tilde{\varphi}_c(x)$ in Example 2) in [I] with T = R belongs to $(S)^*$.

We are now ready to discuss the causal calculus on the space of generalized white noise functionals.

First we introduce differential operators. Recall that the $\mathcal{S}$-transform which carries $(L^2)^-$-functionals to functionals of ξ, denoted by $U(\xi)$. Assume that the functional $U(\xi)$ associated with $(L)^-$-functional $\varphi(x)$ has Fréchet (functional) derivative denoted by $\frac{\delta U}{\delta \xi(t)}$. If the derivative is a U-functional of some $(L^2)^-$-functional denoted by $\varphi_t'(x)$, then φ is differentiable and we write

$$\varphi_t'(x) = \partial_t \varphi(x). \tag{2.4}$$

Formally we write

$$\partial_t = \mathcal{S}^{-1} \frac{\delta U}{\delta \xi(t)} \mathcal{S}, \qquad t \in \mathbb{R}. \tag{2.4'}$$

Remark. Since $\{x(t);\ t \in \mathbb{R}\}$ is taken to be the system of variables of white noise functionals, it is reasonable to introduce an operator $\frac{\partial}{\partial x(t)}$. Indeed, the ∂_t just defined above is a realization of $\frac{\partial}{\partial x(t)}$.

We note that the domain of ∂_t includes $(L)^+$, (S), and the normal functionals with continuous kernels.

Example. Let $\varphi(x)$ be a normal functional with the U-functional of the form

$$U(\xi) = \int_{\mathbb{R}^k} F(u_1,\dots,u_k)\xi(u_1)^{n_1}\dots\xi(u_k)^{n_k}du_1\dots du_k . \tag{2.5}$$

where F is continuous. Then, φ is differentiable and the derivative $\partial_t\varphi$ has U-functional of the form

$$\sum_j n_j\xi(t)^{n_j-1}\int_{\mathbb{R}^{k-1}} F(\dots,\overset{j}{t},\dots)\xi(u_1)^{n_1}..j..\xi(u_k)^{n_k}du_1..j..du_k .$$

We have tacitly proved that the assertion mentioned above that normal functionals are in the domain $\mathfrak{D}(\partial_t)$.

The following assertion is easily proved by applying ∂_t to the exponential functions $\exp[c\langle x,\xi\rangle]$ that has been observed in the Example 1°) of [II].

Proposition 2.3. The differential operator ∂_t is a derivation.

As was introduced in [8], we can define the adjoint operator ∂_t^* for ∂_t in such a way that

$$\langle\varphi,\partial_t f\rangle_\mu = \langle\partial_t^*\varphi, f\rangle_\mu$$

where f is a test functional and φ is a generalized functional.

Theorem 2.1. i) ∂_t is an annihilation operator. In particular

$$\partial_t : \mathcal{H}_n^{(n)} \longrightarrow \mathcal{H}_{n-1}^{(-n-1)},$$

and ∂_t is a continuous map from (S) into itself.

ii) ∂_t^* is a creation operator, in particular

$$\partial_t^*: \mathcal{H}_n^{(-n)} \longrightarrow \mathcal{H}_{n+1}^{(n-1)},$$

and ∂_t^* is a continuous map from $(S)^*$ into itself.

<u>Proof</u>. We use the integral representation and observe that for the kernel F of the $\mathcal{H}_n^{(n)}$-functional

$$F(u_1,\ldots,u_n) \longrightarrow nF(u_1,\ldots,u_{n-1},t)$$

by ∂_t. While, for $\mathcal{H}_n^{(-n)}$-functional

$$G(u_1,\ldots,u_n) \longrightarrow (\delta \hat{\otimes} G)(u_1,\ldots,u_{n+1}).$$

Note that G is a generalized function.

<u>Proposition 2.4</u>. The canonical commutation relation holds:

$$[\partial_t,\partial_s^*] = \delta(t-s)$$

$$[\partial_t,\partial_s] = [\partial_t^*,\partial_s^*] = 0.$$

Multiplication by x(t) is well-defined in such a way that

$$x(t)\cdot = \partial_t^* + \partial_t \ . \tag{2.6}$$

By using the creation operators ∂_t^*, we define a stochastic integral in the generalized sense. For f in $L^2(\mathbb{R}^t)$ we set

$$\partial^*(f) = \int f(u)\partial_u^* du.$$

It is an operator acting on $(L^2)^-$. Let φ be a member of $(L^2)^-$. Then

$$\partial^*(f)\varphi = \int f(u)\partial_u^*\varphi du \tag{2.7}$$

is a generalized stochastic integral.

Note that if $\varphi = 1$, then $\partial^*(f)1$ is nothing but a Wiener integral. If φu is a functional depending on u and if it is non-anticipating, then the integral

$$\int f(u)\partial_u^*\varphi \, du \tag{2.8}$$

is also defined and it is in agreement with the Itô integral.

Theorem 2.2. Let φ be an $(L^2)^-$-functional. Assume that the sequence $\{c_n\}$ defining $(L^2)^-$ satisfies the inequality

$$c_{n+1}^2 > (n+1)c_n^2.$$

Then, φ is in the domain of $\partial^*(f)$ and $\partial^*_{\bullet}(f)\varphi$ is again a member of $(L^2)^-$.

Proof is obtained by evaluating the Sobolev norms of the tensor products of f and kernels of $\mathscr{H}_n^{(-n)}$-components. (See [6].)

§3. Rotation group and harmonic analysis.

As was explained in Section 1, we expect that the infinite dimensional rotation group would shed light on our white noise analysis, in particular on the causal calculus on the space of generalized white noise functionals.

The group O(E) itself is quite big; indeed, it is neither compact nor locally compact. We shall therefore take suitable subgroups and observe relations, like hidden symmetry, with the corresponding calculus.

[I] Finite dimensional rotations.

Take a finite, say n, dimensional subspace E_n of E. If the restriction of a rotation $g \in O(E)$ to $E_n^{\perp}$ is the identity, then g is viewed as an n-dimensional rotation. Hence, it is easy to see that O(E) has a subgroup G_n isomorphic to

$SO(n)$. To make the discussion consistent in n, we may start with a choice of a CONS. $\{\xi_n\}$ and form n-dimensional subspace E_n which is increasing in n. In concordance with E_n is a sequence of subgroups G_n of $O(E)$ isomorphic to $SO(n)$. The inductive limit

$$\varinjlim_n G_n = G_\infty \tag{3.1}$$

involves finite dimensional rotations based on $\{\xi_n\}$.

The infinite dimensional Laplace-Beltrami operator Δ_∞ can be characterized in terms of G_∞, and using $\{\xi_n\}$ it has the following expression:

$$\Delta_\infty = \sum_n \left[\frac{\partial^2}{\partial\xi_n^2} - \langle x, \xi_n\rangle \frac{\partial}{\partial\xi_n} \right]. \tag{3.2}$$

As is well known, the subspace $\mathcal{H}_n$ of (L^2) is the eigenspace of Δ_∞ belonging to the eigenvalue $-n$. Irreducible unitary representation of G_∞ is given on the space $\mathcal{H}_n$.

The operator Δ_∞ can be expressed in terms of the ∂_t and ∂_t^*.

<u>Proposition 3.1</u>. We have

$$\Delta_\infty = -\int \partial_t^* \partial_t \, dt. \tag{3.3}$$

<u>Proof</u>. If we apply $-\int \partial_t^* \partial_t \, dt$ to any φ in $\mathcal{H}_n$, we must have $-n\varphi$. So the operator can be extended to a self-adjoint operator with domain $\mathcal{D}(\Delta_\infty)$.

Analogous to the case of two dimensional rotation, multiplication and the differential operators ∂_t can define the infinitesimal rotation in the following manner: Noting (2.5), we set

$$\begin{aligned} \gamma_{t,s} &= (\partial_t^* + \partial_t)\partial_s - (\partial_s^* + \partial_s)\partial_t \\ &= \partial_t^*\partial_s - \partial_s^*\partial_t. \end{aligned}$$

which is defined to be the infinitesimal generator of the rotation. The generators $\varphi_{t,s}$, $s,t \in R$, characterize the operator Δ_∞ in the following manner. Set

$$B = \iint G(u,v)\partial_n^* \partial_v \, dudv \tag{3.4}$$

where $G(u,v)$ is an $\mathcal{H}^{-3/2}(R^2)$-function. It is easy to see that B is defined on $(L^2)^+$ and is symmetric.

Theorem 3.1. If the operator given by (3.4) commutes with all generators $\gamma_{t,s}$, $s,t \in R$, then B is the infinite dimensional Laplace-Beltrami operator up to a constant.

Proof. The commutator $[B,\gamma_{t,s}] = B\,\gamma_{t,s} - \gamma_{t,s}B$ is easily computed and is given by

$$\partial_s \int G(u,t)\partial_u^* du - \partial_t \int (u,s)\partial_u^* du = \partial_t^* \int G(s,v)\partial_v dv - \partial_s^* \int G(t,v)\partial_v dv.$$

Applying the above operators to exponential functions with the $\mathcal{S}$-transform $\exp[\langle f,\xi\rangle]$, we obtain

$$f(v)\int G(t,u)\xi(u)dt - f(u)\int G(t,v)\xi(t)dt = \xi(u)\int G(v,s)f(s)ds - \xi(v)\int G(u,s)f(s)ds.$$

In the above expression, f and ξ can be taken arbitrary, so that we set $f = \xi$ to obtain

$$f(v)\int G(u,s)f(s)ds = f(u)\int G(v,s)f(s)ds.$$

This implies that the generalized function $G(u,v)$ has to of the form (3.3).

We have now established the following diagram

$$\begin{tikzcd} & \bigoplus_n \mathcal{H}_n \text{ (Fock space)} & \\ G_\infty \arrow[ur] \arrow[rr] & & \Delta_\infty \arrow[ul] \end{tikzcd}$$

Figure 1.

[II] The Lévy group.

Again we fix a CONS $\{\xi_n\}$. Let π be an automorphism of the positive integers. Then, a transformation g_π acting on E is defined in such a way that for

$\xi = \Sigma a_n \xi_n$

$$g_\pi \xi = \Sigma a_n \xi_{\pi(n)} \ .$$

Let Π be the collection of automorphisms π such that

$$(*) \qquad \lim_{N\to\infty} \frac{1}{N} \#\{n \leq N;\ \pi(n) > N\} = 0$$

holds, where #( ) means the cardinal number of integers in the { }. Now set

$$\mathcal{G} = \{g_x;\ \pi \in \Pi,\ g_\pi \in O(E)\}.$$

Obviously $\mathcal{G}$ forms a subgroup of O(E), and it is called the <u>Lévy group</u> (see [1], Part III.).

On the other hand, Lévy has defined the following Laplacian:

$$(3.5) \qquad \Delta_L = \lim_{N\to\infty} \frac{1}{N} \sum_1^N \frac{\partial^2}{\partial \xi_n^2} \ .$$

We call Δ_L the <u>Lévy Laplacian</u>.

As we did in the case of the infinite dimensional Laplace-Beltrami operator, we are given the following formal expression

(3.6) $$\Delta_L = \int(\partial_t)^2(dt)^2, \quad \text{(H.-H. Kuo).}$$

We can, of course, give a good interpretation to the above formula. However, for computational convenience we define the Lévy Laplacian as follows:

(3.7) $$\Delta_L = \int \mathcal{S}^{-1} \frac{\delta^2}{\delta\xi(t)^2} \mathcal{S}\, dt.$$

Remark 1. Note that there are two different kinds of second order functional derivatives; one is $\frac{\delta^2}{\delta\xi(t)\delta\xi(s)}$ and the other is $\frac{\delta^2}{\delta\xi(t)^2}$. For the definition of Δ_L we only use the second one. The first one leads to the Volterra Laplacian.

The following theorem is straightforward.

Theorem 3.2. i) The operator Δ_L commutes with the Lévy group.

ii) The domain of Δ_L involves normal functionals.

iii) The operator Δ_L annihilates (L^2)-functionals.

Remark 2. Two Laplacians Δ_∞ and Δ_L share their roles; the former governs the harmonic analysis arising from G_∞, while the latter acts effectively on the space of generalized functionals having a close connection with the Lévy group.

[III] Whiskers.

We then come to the third subgroup of O(E). A one-parameter subgroup $\{g_t\}$ of O(E) is called a whisker if each g_t comes from a diffeomorphism of the parameter set $\overline{R} = R' \cup \{\infty\}$. It is defined in such a way that

(3.8) $$(g_t\xi)(u) = \xi(\psi_t(u))\sqrt{|\psi_t'(u)|}$$

with a suitable choice of a family $\{\psi_t(u), -\infty < t < \infty\}$ of functions of u satisfying

(3.9) $$\psi_t \cdot \psi_s = \psi_{t+s}.$$

Such a g_t can not, in general, be approximated by finite dimensional rotations under the usual topology.

The most important and in fact the simplest example of a whisker is the shift $\{S_t;\ t \in R^1\}$ defined by

$$(3.10) \qquad (S_t \xi)(u) = \xi(u - t), \qquad t \in R^1.$$

Recalling that u is the time variable we see that the shift stands for propagation of time.

It is known (see [3] Chapter 5) that there are two other simple and important whiskers and that together with the shift they form a three dimensional subgroup G_P of O(E) which is isomorphic to the group PSL(2,R). The group G_P is particularly interesting in probability theory; for one think G_P describes Lévy's projective invariance of Brownian motion. Note that the basic nuclear space should be taken suitably in this case.

§4. Applications to Physics.

Needless to say, there are many applications of white noise analysis, but we are going to explain here only two applications to quantum dynamics.

1). Feynman integrals

We shall give a reformulation of the path integral for the propagator in quantum mechanics in terms of generalized white noise functionals, where the average over possible paths is understood as an expectation over the paths interfered with by Brownian motion. In this sense, our method may be considered to be in line with the idea proposed by Feynman in 1948. Moreover it may be worthwhile to mention that we use generalized functionals instead of a limiting procedure.

Let a Lagrangian L be given:

$$(4.1) \qquad L(\dot{y}, y) = \frac{1}{2} m\dot{y}^2 - V(y),$$

where $V(y)$ is assumed to be smooth enough, non-negative and to grow at most of quadratic order. As is well known, the quantum mechanical transition amplitudes can be thought of as an average over fluctuating paths weighted with an exponential of the classical action

$$A(y) = \int_0^t L(\dot{y},s),\ y(s))ds. \tag{4.2}$$

We are now in a position to choose possible trajectories. We propose that y consists of a sure path y_0 determined uniquely by classical mechanics and a Brownian fluctuation denoted by $B(s)$. Hence, y has to be of the form

$$y(s) = y_0(s) + (\hbar/m)^{1/2} B(s)\ , \qquad 0 \leq s \leq t. \tag{4.3}$$

The choice of the constant in front of $B(s)$ is suggested by the dimension calculus. With this expression of y the propatator is given by the formula

$$G(y_1,y_2,t) = E\{\mathcal{N} \exp[\frac{i}{\hbar}\frac{m}{2}\int_0^t \dot{y}(s)^2 ds + \frac{1}{2}\int_0^t \dot{B}(s)^2 ds] \times \exp[-\frac{i}{\hbar}\int_0^t V(y(s))ds]\delta(y(t)-y_2)\}\ . \tag{4.4}$$

In this expression the action is certainly involved, and in addition we include the second integral so that the measure μ of white noise is made flat. The delta function serves to pin the trajectories to y at time t (concerning the use of this factor, see [7]). Finally, it is noted that the factor $\mathcal{N}$ is necessary to have multiplicative renormalization, necessitated by the term involving $\dot{B}(s)^2$

Examples like free particle, harmonic oscillator and some other cases of known potentials allow us to find actual formulae to see that there is nice agreement with the standard results.

Our idea to reformulate the Feynman integral can be generalized to various cases. Very fruitful results have been obtained by de Falco and D.C.

Khandekar.

2) Dirichlet forms

A brief discussion of the application to a representation of the free massive relativistic scalar boson field is going to be presented.

First we provide an important notion. Set

$$(S)_{>} = \{F \in (S);\ \ F(x) \geq 0 \quad \text{a.e.}\}$$

and define

$$(S)^{*}_{>} = \{\ \varphi \in (S)^{*};\ \langle\varphi, F\rangle_{\mu} \geq 0 \quad \text{for every } F \in (S)_{>}\ \}.$$

A functional in $(S)^{*}_{>}$ is called a positive generalized functional. Note that a member of $(S)^{*}_{>}$ is in general renormalized, so that positivity is not a simple notion (See [10]).

Theorem 4.1. (Y. Yokoi) For φ in $(S)^{*}_{>}$ there exists uniquely a probability measure υ_{φ} on E^{*} such that

$$\langle\varphi, F\rangle_{\mu} = \int_{E^{*}} \tilde{F}(x)\,d\upsilon_{\varphi}(x), \qquad F \in (S), \tag{4.5}$$

where $\tilde{F}(x)$ is a continuous version of F.

Define the gradient operator ∇:

$$(\nabla F) = (\partial_t F;\ t \in R^1)$$

and denote

$$|\nabla F|^2 = \int |\partial_t F|^2\, dt.$$

Introduce the Hilbert space $(\mathcal{L}^2) = (L^2) \otimes L^2(R^1)$.

Proposition 4.1. i) ∇ maps (S) into $(\mathcal{L}^2)$.
ii) $|\nabla F|^2$ is in (S) for any $F \in (S)$. With this background a bilinear form is

introduced:

$$(4.6) \qquad \mathcal{E}(F,F) = \langle \varphi, \nabla\overline{F} \cdot \nabla G \rangle = \int (\nabla\overline{F} \cdot \nabla G)(x)\varphi(x)d\mu(x), \qquad \varphi \in (S)^*_> ,$$

where $(\nabla\overline{F} \cdot \nabla G)(x) = \int (\partial_t \overline{F})(x)(\partial_t G)(x)dt$.

We are now interested in the closability of $\mathcal{E}$.

The following theorem is our main result.

Theorem 4.2. If $\varphi \in (S)^*_>$ is such that $\partial_s \varphi = B(s)\varphi$ for every s with $\int B(s)\eta(s)ds \in (S)$ for every $\eta \in \mathcal{S}(R^1)$, then the $\mathcal{E}$ is closable.

For proof we use the well known Kato theorem on closability and several basic properties of the test functional space (S).

Further developments have been made by Albeverio, Potthoff, Röckner, Streit and the present author.

§5. Gaussian random fields

In this section the author wishes to propose a new method of study of Gaussian random fields using the variational calculus. It is difficult to describe the whole story including motivations, background and ideas of the proofs of theorems, however the route of our approach will be illustrated step by step.

1°) Typical examples

i) The Lévy Brownian motion (1945) $\{X(t);\ t \in F^d\}$ is a Gaussian system with $EX(t) = 0$

$$(5.1) \qquad \Gamma(t,s) = E\{X(t)X(s)\} = \frac{1}{2}(|t| = |s| - |t-s|).$$

ii) The Ornstein-Uhlenbeck field $\{U_m(t);\ t \in R^d\}$ is a generalized stationary Gaussian random field with characteristic functional

$$(5.2) \qquad C_m(\xi) = \exp[-\frac{1}{2}\int_{R^d} \frac{|\hat{\xi}(\lambda)|^2}{m^2 + |\lambda|^2} dx], \qquad \xi \in \mathcal{S}(R^d)$$

where $\hat{\xi}$ is the Fourier transform of ξ.

iii) Let $\{X(t);\ t \in R^2\}$ be a Lévy Brownian motion. Set

$$(5.3) \qquad Y(C) = E(X(t)/X(s),\ s \in C),$$

where C is a contour in R^2 and where t is fixed. Then, we have a random field $\{Y(C)\}$ depending on a contour in a plane.

Our aim is to investigate the way of dependency when t changes or when C moves, deforms or is distorted.

2°) The Lévy Brownian motion.

Among others the Lévy Brownian motion is a most interesting field. Let us assume d = 2 to fix the idea. If the parameter t is restricted to a C^∞-curve C, then using the arc length we are given a Gaussian process depending on a one-dimensional parameter. The most interesting example of C, except for a straight line, is a circle. We may assume that the circle C is originated from the origin. We are given a Gaussian process $\{X(\theta)\}$, the canonical representation (in the Lévy sense) of which is given by

$$(5.4) \qquad X(\theta) = \int_0^\theta \{\sin\theta(\csc\frac{\theta'}{2} - \frac{\cot\frac{\theta'}{4}}{2} h(\theta')) + \cos^2\frac{\theta}{4} h(\theta')\} dB(\theta'),$$

where $h(\theta) = \{1 + \frac{\theta}{4}\tan\frac{\theta}{4}\}^{-1}$ (Si Si, 1989, see [11]). This representation tells us that $\{X(\theta)\}$ is a double Markov Gaussian process.

Here is a conjecture: There is no smooth curve C such that a Gaussian process with parameter set C is a finite order Markov process except the case of constant curvature.

We then come to conditional expectations as in (5.3). Let C be a circle.

The values t, x, ρ, θ and the point p are as in Figure 2.

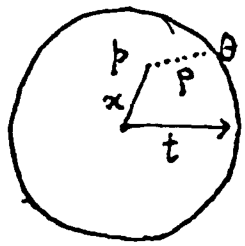

Figure 2.

Then the conditional expectation

$$Y(C) = E(X(p)/X(\theta),\ \theta \in C)$$

is given by the following formula (also in [11]).

$$Y(C) = \int_0^{2\pi} f(p,\theta) X(\theta) d\theta, \tag{5.5}$$

where

$$f(p,\theta) = \frac{(t^2-x^2)^2}{8t\rho^3} + \frac{1}{2\pi}\left(1 - \frac{t+x}{2t} E\left(\frac{\pi}{2}, \frac{2\sqrt{tx}}{t+x}\right)\right),$$

(E: elliptic function.)

Remark. The formula (5.5) gives us lots of suggestions. For instance, the factor ρ^3 in the first term of f is one of the characteristics of the Lévy Brownian motion. If we know $X(\theta)$ only on part of C, then f must involve a generalized function which can be shown by the canonical representation thoery.

2°) White noise with higher dimensional parameter

Start with a Gel'fand triple

$$E \subset L^2(R^d) \subset E^*$$

and introduce white noise measure μ on E^*. As in Section 1, we form a Hilbert space $(L^2) = L^2(E^*, \mu)$. Take the subspace $\mathcal{H}$, and form $\mathcal{H}_1^{(-1)}$ consisting of generalized linear functionals of $x \in E^*$. The U-functional associated with $\mathcal{H}_1^{(-1)}$-functionals is expressible as

$$U(\xi) = \langle f, \xi \rangle, \qquad f \in H^{-(d+1)/2}(R^d). \tag{5.6}$$

With the help of such an expression, one can speak of the support of f. We can consider only those $\mathcal{H}_1^{(-1)}$-functionals for which kernels are supported by a lower dimensional manifold. This means that we can restrict the parameter of white noise to a manifold in R^d.

4°) <u>Random fields depending on C and their variations</u>.

Let $\mathbb{C}$ be a class of C^∞-curves homeomorphic to a circle, and consider a Gaussian random field $\{X(C): C \in \mathbb{C}\}$

<u>Case 1</u>. This is the simplest case. Set

$$X(C) = \int_{[C]} f(u)X(u)du, \qquad f \in H^{3/2}(R^2)$$

where [C] is the domain with boundary C. Then, the variation is

$$\delta X(C) = \int_C f(x)x(s)\delta n(s)ds,$$

which implies

$$X_n'(C)(s) = f(s)x(s).$$

We can therefore recover x(s) where f does not vanish.

<u>Case 2</u>. More generally, we set

$$X(C) = \int_{[C]} f(C,u)x(u)du.$$

Then

$$\delta X(C) = \int_{[C]} \delta f(C,u)x(u)du + \int_C f(C,s)x(s)\delta n(s)ds.$$

The two integrals above have different order in the mean square sense. So, they can be discriminated, i.e; x(s) can be recovered under suitable assumptions on f.

Case 3. White noise integral over a curve. We set

$$Y(C) = \int_C f(C,s)X(s)ds,$$

where X is taken to be an ordinary Gaussian random field with parameter set R^2.

Then

$$\delta Y(C) = \int_C \{\delta f(C,s) - kf(C,s)\delta n(s)\}X(s)ds + \int_C f(C,s) \frac{\partial}{\partial n} X(s)\delta n(s)ds$$

where k is the curvature. Actual examples can be seen in [12]. Note that the curvature appears by the variation of the line element ds.

Case 4. The case where $\mathbb{C}$ is taken to be the set of plane circles. Variation should be taken within $\mathbb{C}$. We do not want to go into details, but the subgroup of the third kind (see Section 3, [III]) plays an important role and we can even appeal to the unitary representation theory of Lie groups (see [13]).

Case 5. We still consider the case of R^2-parameter white noise x. Let D be a domain with boundary $C = \partial D$ in $\mathbb{C}$. Define

$$X(t,C) = \int_D G(t,s;C)x(s)ds,$$

where G is the Green's function. Obviously, for C fixed we have

$$\Delta_t X(t,C) = x(t),$$

so X is now an "innovation".

Then, letting t be fixed, we take the variation in C.

$$\delta X(t,C) = \int_D \delta G(t,s:C)x(s)ds + \int_C G(t,s:C)x(s)\delta n(s)ds.$$

To discuss this variation, we can use the famous Hadamard equation:

$$\delta G = \frac{-1}{2\pi} \int_C \frac{\partial}{\partial n} G(u,m;C) \frac{\partial}{\partial n} G(m,u;C)\delta n(s)ds, \qquad m = m(s).$$

For further discussions see [14].

References

[1] P. Lévy, Problèmes concrets d'analyse fonctionnelle, Gauthier-Villars, 1951.

[2] G. Kallianpur and R.L. Karandikar, White noise theory of prediction, filtering and smoothing. Gordon & Breach Sci. Pub. 1988.

[3] T. Hida, Brownian motion. Iwanami 1975; Springer-Verlag 1980.

[4] T. Hida, Analysis of Brownian functionals. Carleton Univ. Math. Lec. Notes #13, 1975.

[5] T. Hida, H.-H. Kuo, J. Potthoff and L. Streit, White noise: an infinite dimensional calculus, monograph in preparation.

[6] T. Hida, Generalized Brownian functionals and stochastic integrals. Appl. Math. and Optimization 12 (1989), 115-123.

[7] T. Hida and L. Streit. Generalized Brownian functionals, Lecture Notes in Physics #153 (1981) 285-287.

[8] I. Kubo and S. Takenaka, Calculus on Gaussian white noise. I-IV; Proc. Japan Acad. 56-58 (1980-1982).

[9] T. Hida, J. Potthoff and L. Streit, Dirichlet forms and white noise analysis. Commun. Math. Phys. 116 (1988) 235-245.

[10] Y. Yokoi, Positive generalized functionals. Preprint. (1987).

[11] Si Si, A note on Lévy's Brownian motion. Nagoya Math. J. 108 (1987), 121-130.

[12] Si Si, Gaussian processes and conditional expectations, BiBoS Notes Nr. 292/87. Universität Bielefeld, 1987.

[13] K.-S. Lee, White noise approach to Gaussian random fields, to appear.

[14] T. Hida and Si Si, Variational calculus for Gaussian random fields. Proc. 1988 IFIP Warsaw Conference.

[15] H.-H. Kuo, Brownian functionals and applications. Acta Appl. Math. 1 (1983), 175-188.

219. J. Bather. Stopping rules and ordered families of distributions. Dec. 87. *Sequential Anal.*, 7, 1988, 111-126.

220. S. Cambanis and M. Maejima. Two classes of self-similar stable processes with stationary increments. Jan. 88. *Stochastic Proc. Appl.*, 32, 1989, 305-329.

221. H.P. Hucke, G. Kallianpur and R.L. Karandikar. Smoothness properties of the conditional expectation in finitely additive white noise filtering. Jan. 88. *J. Multivariate Anal.*, 27, 1988, 261-269.

222. I. Mitoma. Weak solution of the Langevin equation on a generalized functional space. Feb. 88. (Revised as No. 238).

223. L. de Haan, S.I. Resnick, H. Rootzén and C. de Vries. Extremal behaviour of solutions to a stochastic difference equation with applications to arch-processes. Feb. 88. *Stochastic Proc. Appl.*, 32, 1989, 213-224.

224. O. Kallenberg and J. Szulga. Multiple integration with respect to Poisson and Lévy processes. Feb. 88. *Prob. Theor. Rel. Fields*, 83, 1989, 101-134.

225. D.A. Dawson and L.G. Gorostiza. Generalized solutions of a class of nuclear space valued stochastic evolution equations. Feb. 88. *Appl. Math. Optimization*, to appear.

226. G. Samorodnitsky and J. Szulga. An asymptotic evaluation of the tail of a multiple symmetric α-stable integral. Feb. 88. *Ann. Probability*, 17, 1989, 1503-1520.

227. J.J. Hunter. The computation of stationary distributions of Markov chains through perturbations. Mar. 88.

228. H.C. Ho and T.C. Sun. Limiting distribution of nonlinear vector functions of stationary Gaussian processes. Mar. 88. *Ann. Probability*, to appear.

229. R. Brigola. On functional estimates for ill-posed linear problems. Apr. 88.

230. M.R. Leadbetter and S. Nandagopalan. On exceedance point processes for stationary sequences under mild oscillation restrictions. Apr. 88. *Proc. Oberwolfach Conf. on Extremal Value Theory*, J. Hüsler and R. Reiss, eds., Springer, 1989, 69-80.

231. S. Cambanis, J. P. Nolan and J. Rosinski. On the oscillation of infinitely divisible processes. Apr. 88. *Stochastic Proc. Appl.*, to appear.

232. G. Hardy, G. Kallianpur and S. Ramasubramanian. A nuclear space-valued stochastic differential equation driven by Poisson random measures. Apr. 88.

233. D.J. Daley, T. Rolski. Light traffic approximations in queues (II). May 88. *Math. Operat. Res.*, to appear.

234. G. Kallianpur, I. Mitoma, R.L. Wolpert. Nuclear space-valued diffusion equations. July 88. *Stochastics*, 1989, to appear.

235. S. Cambanis. Admissible translates of stable processes: A survey and some new models. July 88.

236. E. Platen. On a wide range exclusion process in random medium with local jump intensity. Aug. 88.

237. R.L. Smith. A counterexample concerning the extremal index. Aug. 88. *Adv. Appl. Probab.*, 20, 1988, 681-683.

238. G. Kallianpur and I. Mitoma. A Langevin-type stochastic differential equation on a space of generalized functionals. Aug. 88.

239. C. Houdré. Harmonizability, V-boundedness, (2,P)-boundedness of stochastic processes. Aug. 88. *Prob. Th. Rel. Fields*, to appear.

240. G.W. Johnson and G. Kallianpur. Some remarks on Hu and Meyer's paper and infinite dimensional calculus on finitely additive canonical Hilbert space. Sept. 88. *Th. Prob. Appl.*, to appear.

241. L. de Haan. A Brownian bridge connected with extreme values. Sept. 88. *Sankhya*, 1989, to appear.

242. O. Kallenberg. Exchangeable random measures in the plane. Sept. 88. *J. Theor. Probab.*, to appear.

243. E. Masry and S. Cambanis. Trapezoidal Monte Carlo integration. Sept. 88. *SIAM J. Numer. Anal.*, 1989, to appear.

244. L. Pitt. On a problem of H.P. McKean. Sept. 88. *Ann. Probability*, 17, 1989, 1651-1657.

245. C. Houdré. On the linear prediction of multivariate (2,P)-bounded processes. Sept. 88.

246. C. Houdré. Stochastic processes as Fourier integrals and dilation of vector measures. Sept. 88. *Bull. Amer. Math. Soc.*, 21, 1989, 281-285.

247. J. Mijnheer. On the rate of convergence in Strassen's functional law of the iterated logarithm. Sept. 88. *Probab. Theor. Rel. Fields*, to appear.

248. G. Kallianpur and V. Perez-Abreu. Weak convergence of solutions of stochastic evolution equations on nuclear spaces. Oct. 88. *Proc. Trento Conf. on Infinite Dimensional Stochastic Differential Equations*, 1989, to appear.

249. R.L. Smith. Bias and variance approximations for estimators of extreme quantiles. Nov. 88.

250. H.L. Hurd. Spectral coherence of nonstationary and transient stochastic processes. Nov. 88. 4th Annual ASSP Workshop on Spectrum Estimation and Modeling, Minneapolis, 1988, 387-390.

251. J. Leskow. Maximum likelihood estimator for almost periodic stochastic processes models. Dec. 88.

252. M.R. Leadbetter and T. Hsing. Limit theorems for strongly mixing stationary random measures. Jan. 89. *Stochastic Proc. Appl.*, to appear.

253. M.R. Leadbetter, I. Weissman, L. de Haan, H. Rootzén. On clustering of high values in statistically stationary series. Jan. 89. *Proc. 4th Int. Meeting on Stat. Climatology*, 217-222.

254. J. Leskow. Least squares estimation in almost periodic point processes models. Feb. 89.

255. N.N. Vakhania. Orthogonal random vectors and the Hurwitz-Radon-Eckmann theorem. Apr. 89.

256. E. Mayer-Wolf, A central limit theorem in nonlinear filtering, Apr. 89.

257. C. Houdré, Factorization algorithms and non-stationary Wiener filtering, Apr. 89. Stochastics, to appear.

258. C. Houdré, Linear Fourier and stochastic analysis, Apr. 89.

259. G. Kallianpur, A line grid method in areal sampling and its connection with some early work of H. Robbins, Apr. 89. *Amer. J. Math. Manag. Sci.*, 1989, to appear.

260. G. Kallianpur, A.G. Miamee and H. Niemi, On the prediction theory of two-parameter stationary random fields, Apr. 89. *J. Multivariate Anal.*, to appear.

261. I. Herbst and L. Pitt, *Diffusion equation techniques* in stochastic monotonicity and positive correlations, Apr. 89.

262. R. Selukar, On estimation of Hilbert space valued parameters, Apr. 89. (*Dissertation*)

263. E. Mayer-Wolf, The noncontinuity of the *inverse Radon transform* with an application to probability laws, Apr 89.

264. D. Monrad and W. Philipp, Approximation theorems for weakly dependent random vectors and Hilbert space valued martingales, Apr. 89.

265. K. Benhenni and S. Cambanis, Sampling designs for estimating integrals of stochastic processes, Apr. 89.

266 S. Evans, Association and random measures, May 89.

267. H.L. Hurd, Correlation theory of almost periodically correlated processes, June 89.

268. O. Kallenberg, Random time change and an integral representation for marked stopping times, June 89. *Probab. Th. Rel. Fields*, accepted.

269. O. Kallenberg, Some uses of point processes in multiple stochastic integration, Aug. 89.

270. W. Wu and S. Cambanis, Conditional variance of symmetric stable variables, Sept. 89.

271. J. Mijnheer, U-statistics and double stable integrals, Sept. 89.

272. O. Kallenberg, On an independence criterion for multiple Wiener integrals, Sept. 89.

273. G. Kallianpur, *Infinite dimensional* stochastic differential equations with applications, Sept. 89.

274. G.W. Johnson and G. Kallianpur, Homogeneous chaos, p-forms, scaling and the Feynman integral, Sept. 89.

275. T. Hida, A white noise theory of infinite dimensional calculus, Oct. 89.

276. K. Benhenni, Sample designs for estimating integrals of stochastic processes, Oct. 89. (Dissertation)

277. I. Rychlik, The two-barrier problem for continuously differentiable processes, Oct. 89.

278. G. Kallianpur and R. Selukar, Estimation of Hilbert space valued parameters by the method of sieves, Oct. 89.

279. G. Kallianpur and R. Selukar, Parameter estimation in linear filtering, Oct. 89.

280. P. Bloomfield and H.L. Hurd, Periodic correlation in stratospheric ozone time series, Oct. 89.

281. J.M. Anderson, J. Horowitz and L.D. Pitt, On the existence of local times: a geometric study, Jan. 90.

282. G. Lindgren and I. Rychlik, Slepian models and regression approximations in crossing and extreme value theory, Jan. 90.

283. H.L. Koul, M-estimators in linear models with long range dependent errors, Feb. 90.

284. H.L. Hurd, Almost periodically unitary stochastic processes, Feb. 90.